

Magnon bound-state scattering in gauge and string theory

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ABSTRACT: It has been shown that, in the infinite length limit, the magnons of the gauge theory spin chain can form bound states carrying one finite and one strictly infinite R-charge. These bound states have been argued to be associated to simple poles of the multi-particle scattering matrix and to world sheet solitons carrying the same charges. Classically, they can be mapped to the solitons of the complex sine-Gordon theory. Under relatively general assumptions we derive the condition that simple poles of the two-particle scattering matrix correspond to physical bound states and construct higher bound states “one magnon at a time”. We construct the scattering matrix of the bound states of the BDS and the AFS S-matrices. The bound state S-matrix exhibits simple and double poles and thus its analytic structure is much richer than that of the elementary magnon S-matrix. We also discuss the bound states appearing in larger sectors and their S-matrices. The large 't Hooft coupling limit of the scattering phase of the bound states in the SU(2) sector is found to agree with the semiclassical scattering of world sheet solitons. Intriguingly, the contribution of the dressing phase has an independent world sheet interpretation as the soliton-antisoliton scattering phase shift. The small momentum limit provides independent tests of these identifications.

KEYWORDS: AdS-CFT Correspondence, Integrable Field Theories, Gauge-gravity correspondence.

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1. Introduction

There is mounting evidence that both the spectrum of anomalous dimensions of infinitely long operators in $\mathcal{N} = 4$ super-Yang-Mills theory and the spectrum of the world sheet sigma model (defined on a plane) can be described in terms of Bethe ansätze. They are based on the scattering matrices of the fundamental excitations building, respectively, the gauge theory gauge invariant operators and the physical string states. The same information is encoded in the scattering matrix of the momentum eigenstate presentation of these excitations — the magnon scattering matrix. There currently exists an all-loop conjecture for the scattering of gauge theory magnons [1, 2] as well as indirect results for the leading order [3] and the first subleading correction [4, 5] to the scattering phase of string theory magnons. They have been argued for and tested in detail in the large 't Hooft coupling and small world sheet momentum regime, in which the gauge theory magnons are in one to one correspondence with the world sheet fields in the uniform gauge. The string theory

magnon scattering matrix is further conjectured to have the same expression even if the world sheet momenta are held fixed in the large 't Hooft coupling limit. It is important to subject it to controlled tests in this regime.

More generally, an algebraically-determined scattering matrix needs to be subjected to consistency tests. A distinct possibility is that this S-matrix is unrelated to the lagrangian one would like to quantize. If the states scattered by it are directly related to the fields of the original lagrangian, the details of the S-matrix may be tested by direct perturbative higher order calculation.¹ If the exact S-matrix describes physical bound states one may, in the same spirit, test whether their scattering is correctly reproduced by corrections to the classical scattering induced by the original lagrangian.

In an arbitrary field theory, the scattering of bound states (if they exist) is related in a rather complicated way to the scattering of the fundamental fields. The bootstrap approach was proposed as a way of determining both the spectrum of bound states and their scattering amplitudes. Its main postulates are that

1. the scattering amplitudes are determined self-consistently; all particles that can appear as scattering states are the same particles being exchanged in the scattering process and thus the S-matrix exhibits physical simple poles corresponding the their going on-shell.²
2. the S-matrix is unitary (in a generalized sense) and has prescribed analytic properties.

In two-dimensional relativistic unitary integrable quantum field theories these postulates have been used to great effect to determine exact S-matrices. The calculation of bound state scattering matrices is simplified by the lack of particle production in that it is determined (up to an overall phase) by that of the fundamental excitations. The successful comparison of the bootstrap-constructed bound state S-matrix and the semiclassical lagrangian calculation of the same quantity is a test of the consistency of the S-matrix of the fundamental excitations.

In the context of the AdS/CFT correspondence, it has been shown in [11] that, after relaxing the level matching condition, there exist classical solutions with finite momenta of the world sheet theory whose semiclassical scattering reproduces the large 't Hooft coupling limit of the AFS scattering matrix (with fixed momentum). Solitons of higher charges (dyonic giant magnons) and in larger sectors have been also constructed [12]–[16]. They carry two R-charges. One of them, J_1 , is strictly infinite³ leading to an infinitely-long string. The second charge J_2 corresponds to rotation in the direction orthogonal to J_1 .

¹It is worth mentioning that naive “exact” S-matrices for the non-simply laced affine Toda theories fail this test at the 1-loop level [6, 7]. To a certain extent, the calculation mentioned here explicitly tests the quantum integrability of the theory. These theories are nevertheless integrable at the quantum level. The “true” exact S-matrices for the non-simply laced affine Toda theories were constructed and analyzed in [8–10]. Their consistency with perturbation theory is subtle and interesting. I would like to thank Patrick Dorey for clarifying this to me.

²It is worth emphasizing that it is not necessary that any pole satisfying physical state conditions should correspond to a bound state. They may simply correspond to “fundamental” states which have been missed. A classic example is the appearance of the closed string poles in the open string scattering amplitudes.

³See [17] for a discussion of $1/J_1$ corrections.

The charges of semiclassical solitons are physically unrelated to the 't Hooft coupling. However, since the classical sigma model corresponds to infinite 't Hooft coupling and the second charge J_2 can be (in principle) arbitrary, there are (at least) two natural regimes one may consider: *i*) J_2 is fixed in the large 't Hooft coupling limit or *ii*) J_2 scales with the 't Hooft coupling as suggested by the classical lagrangian: $J_2 \propto \sqrt{\lambda}$. While both situations formally appear as classical solutions of the sigma model, all solutions of the former type differ only by parametrically small terms (suppressed by factors of $1/\sqrt{\lambda}$) and thus it is not completely clear why they should, by themselves, be considered as distinct and trustworthy classical solutions. One way of understanding them is as a small charge limit of solutions of the second type.

Perhaps a more controlled set of states are those whose charges scale as indicated by the classical lagrangian $J \propto \sqrt{\lambda}$. As discussed in [14], they are visible in the finite gap equations describing the SU(2) sector and, apart from exhibiting finite world sheet momentum, appear in the same parameter space as the finite density configurations:

$$\begin{aligned}
 E - L &= \sqrt{J_2^2 + 4\bar{\lambda} \sin^2 \frac{p}{2}} - J_2 \equiv \sqrt{\bar{\lambda}} f(J_2/\sqrt{\bar{\lambda}}) \\
 \mathcal{J}_2 &= J_2/\sqrt{\bar{\lambda}} = \text{fixed} \quad \bar{\lambda} = \frac{\lambda}{4\pi^2}.
 \end{aligned}
 \tag{1.1}$$

The ratio $\mathcal{J}_2 = J_2/\sqrt{\bar{\lambda}}$ can be treated as a free parameter and $L = J_1 + J_2$ is the length of the string (and of the corresponding gauge theory operator). It has been argued [13] that, in the strong coupling limit, both the states with scaling and fixed charges (in the sense emphasized above) correspond to the solitons of the complex sine-Gordon theory (CsG).

In this note we will compare the scattering of semiclassical strings corresponding to magnon bound states as predicted by the conjectured string magnon scattering matrix

$$S_{\text{string}} = S_{\text{gauge}} \sigma^2 \quad \sigma^2 = e^{i\theta_{\text{AFS}}}
 \tag{1.2}$$

and by the classical sigma model S-matrix. Here S_{gauge} denotes the conjectured all-loop gauge theory magnon S-matrix [1, 2]. The “dressing phase” σ was originally constructed [3] to leading order in the large 't Hooft coupling limit by analyzing states with two angular momenta J_1 and J_2 of the same order and with small world sheet momenta. In this regime there is no sharp separation between quantum (i.e. $1/\sqrt{\bar{\lambda}}$) and finite size (i.e. $1/L \sim 1/J$) corrections; σ captures — through the Bethe equations — the analytic part of the one-loop corrections to semiclassical string states. The regime we will be probing is quite different: the angular momentum J_1 is strictly infinite, the world sheet momenta is kept fixed and the second angular momentum is fixed in units of $\sqrt{\bar{\lambda}}$. The fact that the “dressing phase” σ was constructed in a different regime than the one in which the bound states appear makes our comparison nontrivial. Within our setup we will interpret the states with fixed charge as the $J_2/\sqrt{\bar{\lambda}} \rightarrow 0$ limit of $J_2/\sqrt{\bar{\lambda}} = \text{fixed}$ states.

After a brief discussion of bound states in general integrable field theories we proceed in §3 with a review of the construction of bound state S-matrices via the fusion procedure. We will not assume two-dimensional Lorentz invariance, having in mind applications to the gauge and string side of the AdS/CFT correspondence for which the relevant S-matrices do

not exhibit this symmetry. While this makes the original relativistic analysis inapplicable, a weaker set of assumptions (covering both the gauge and string S-matrices) leads to a general condition for the unitarity of the bound state S-matrices. This also leads to a condition that the state corresponding to a simple pole is physical (i.e. that its wave function is normalizable). For certain choices of variables this condition has a simple form.

In §4 we will then apply this procedure to the scattering of bound states described by the BDS S-matrix as well as to the bound states described by the general ansatz for the string scattering matrix. We will find that for finite charge states the scattering phase receives two contributions of the same order — one comes from the dressing phase while the second one is generated by the prefactor S_{gauge} — and discuss whether both should be visible in the world sheet sigma model. We will also notice that the physical state conditions prevent the existence of bound states in some sectors, in agreement with previous studies [14]. In the weak coupling limit we will nevertheless find the partners of the magnon bound states in other sectors by analyzing the $SU(2|2)^2$ -invariant S-matrix.

We will then proceed to reconstruct the scattering phase of the sigma model solitons from that of their CsG counterparts originally discussed in [18, 19] and then to compare the result with the prediction of the fusion construction. We will find that they agree and that there are two ways to interpret this agreement. On the one hand, the contribution of the dressing phase is correctly reproduced by the soliton-*antisoliton* scattering while that of S_{gauge} is interpreted as a coherent superposition of one-loop effects. On the other hand, the sum of the contribution of the dressing phase and of S_{gauge} is correctly reproduced by the soliton-soliton scattering. We independently confirm these identifications in the small momentum limit in §8.

2. Bound states and 2-particle S-matrices

The spectrum of bound states of a field theory typically has a complicated structure. In particular, there can be bound states of two or more particles, perhaps forming various representations of some symmetry group. The fundamental property of an integrable field theory defined on a plane is that all physical information is encoded in the 2-particle scattering matrix for its fundamental excitations.⁴ In particular, all information about bound states of arbitrary charge is encoded in it. While it is relatively clear that two-particle bound states may appear as poles in the two-particle S-matrix, it is also intuitively clear that many-particle bound states cannot directly appear in the fundamental 2-particle S-matrix. Indeed, let us assume that the scattered particles carry a representation of some group and that their scattering matrix respects this symmetry (which may not be the complete symmetry of the S-matrix). It follows then that the bound states that are visible in the two-particle S-matrix are those transforming in (irreducible) representations appearing in the direct product of the representations labeling this S-matrix.

⁴It is typically not *a priori* clear what are the fundamental excitation of a theory and what is their relation to the fields appearing in its lagrangian. There exist examples in which the fields appearing in the lagrangian are bound states of some such “more fundamental” excitations (Toda field theories).

Rather, many-particle bound states are described by the poles of the many-particle S-matrix which, in turn is determined by the 2-particle one. In the context of the AdS/CFT correspondence, multi-magnon bound states and the corresponding world sheet solitons have been identified in [12, 14] in compact sectors of the theory. They are the so-called Bethe strings. From the stand point of the gauge theory spin chain, they represent deformations of the Bethe strings of the Heisenberg chain. The details of both the conjectured gauge and string theory S-matrices imply that, in fact, on the rapidity plane $u(p)$ the structure of the Bethe strings remains the same as for the Heisenberg chain — two rapidities differ by an integer multiple of the imaginary unit. In momentum space their shape changes in a coupling constant dependent way.

While the identification of the Bethe strings in the multi-particle scattering matrix is a perfectly valid standpoint, it is somewhat complicated to extract and justify the properties of these bound states. For this purpose it is perhaps more useful to build bound states “one magnon at a time”.

An interesting question is whether all possible bound states can be identified this way. One may expect that magnon bound states are only part of the total number of bound states. Also, since the magnon scattering matrix is diagonal, it is not immediately clear how the symmetries are realized. We will see this more explicitly in the following sections. More generally, the identification of *all* the bound states of a theory depends on one’s ability to identify the fundamental excitations and determine their scattering matrix. This is relatively clear by considering the manifest symmetry group of the scattering matrix. The fundamental excitations as well as their bound states form linear representations of this group. Then, up to accidental zeros of the S-matrix, the corresponding Clebsch-Gordan coefficients determine which bound states are formed. Consequently, to identify all possible bound states it is necessary to identify the excitations carrying the smallest possible representation.⁵

3. Bootstrap and fusion

A very successful application of the bootstrap and fusion ideas is in the context of integrable field theories, where they have been successfully used to derive the scattering matrices of bound states. In certain cases the result was successfully justified by other means. We will review here the fusion construction [20]–[22] being however careful not to assume that the scattering matrix is relativistically invariant and derive the conditions that bound states are physical.

Let us assume that for some values of rapidities

$$u_1 = a_1 + ib_1 \quad u_2 = a_2 + ib_2 \tag{3.1}$$

⁵For bosonic groups this is the fundamental representation. It is however important to emphasize that for supergroups the smallest representation may in fact be unphysical. Nevertheless, the scattering of the excitations carrying this (unphysical) representation determines the S-matrices of the physical bound states. Alternatively, in the space of physical states it is necessary to have the fundamental excitations transforming in a reducible representation of the (perhaps nonlinearly-realized) symmetry group.

the fundamental S-matrix S_{12} has a simple pole with residue R_{12} . The precise values of (a_1, a_2, b_1, b_2) depend on the details of S_{12} . This residue may be decomposed in projectors onto representations of the manifest symmetry group of the S-matrix:

$$R_{12} = \sum_a R_{12}^a P_a. \tag{3.2}$$

It is worth pointing out that, if the S-matrix has a larger symmetry than that which is manifest, the various representations appearing in (3.2) should realize (perhaps nonlinearly) this larger symmetry group.

From here it is natural to infer that the scattering matrix of the (12) bound state with the particle 3 is proportional to the residue of the pole at the position (3.1) of $S_{123} = S_{12}S_{13}S_{23}$. In general this residue does not satisfy the Yang-Baxter equation or unitarity. The ansatz for the S-matrix is

$$S_{(12)3} = A \text{Res}_{(12)}(S_{12}S_{13}S_{23})B = AR_{12}S_{\hat{1}3}S_{\hat{2}3}B \tag{3.3}$$

with A and B determined by unitarity and factorization. The hatted indices denote the fact that the corresponding spectral parameters are evaluated on (3.1).

From the Yang-Baxter equation it is trivial to read that

$$R_{12}S_{\hat{1}3}S_{\hat{2}3} = S_{\hat{2}3}S_{\hat{1}3}R_{12} \tag{3.4}$$

while projecting from the left and from the right this equation onto $\sum_a P_a$ and $1 - \sum_a P_a$ we find

$$(1 - \sum_a P_a) S_{\hat{2}3}S_{\hat{1}3}R_{12} \sum_b P_b = 0. \tag{3.5}$$

Requiring that the Yang-Baxter equation is satisfied by the ansatz (3.3) implies that, up to an overall function, A and B must satisfy

$$BAR_{12} = \sum_a P_a. \tag{3.6}$$

The calculation is the same as in a relativistic field theory.

Relativistic invariance makes it easy to extract general information from the requirement that the S-matrix (3.3) is unitary

$$S_{(12)3}^\dagger S_{(12)3} = 1. \tag{3.7}$$

We are however interested in more general situations, such as the S-matrices appearing on the gauge theory or on the string theory side of the AdS/CFT correspondence. It turns out that the various conjectured S-matrices obey the following identity

$$\begin{aligned} S_{\hat{1}3}(u_1, u_3)^\dagger &= S_{\hat{1}3}(u_3, u_1^*) = S_{\hat{1}3}(u_3, u_2) \\ &= E_{12}S_{\hat{2}3}(u_3, u_2)E_{12} = E_{12}S_{\hat{2}3}(u_2, u_3)^{-1}E_{12} \end{aligned} \tag{3.8}$$

where E_{12} is the operator switching the labels of states 1 and 2 while leaving their momenta unchanged:

$$E_{12}|\alpha(1)\beta(2)\rangle = |\beta(1)\alpha(2)\rangle. \quad (3.9)$$

We also used the fact that the bound state (12) appears if the u -parameters of the states 1 and 2 are complex conjugates of each other. Using the identity (3.8) it is not hard to show that (3.7) is satisfied if

$$E_{12}R_{12}^\dagger A^\dagger A = \mathcal{C} \sum_a P_a, \quad (3.10)$$

where \mathcal{C} is an arbitrary function. Besides determining A this condition identifies which poles are physical and which are not. Consider acting with (3.10) on an eigenstate in a definite representation and let us denote by η_a the eigenvalue of the operator E_{12} corresponding to this state. The operator $A^\dagger A$ is positive definite. Using the spectral decomposition (3.2) of the residue R_{12} it follows that

$$\text{sgn}(\mathcal{C}) \eta_a R_{12}^a > 0. \quad (3.11)$$

There is still an ambiguity due to the unknown sign of \mathcal{C} . However, the important point is that the same function \mathcal{C} appears for all states and thus we may determine its sign from one state. For relativistic field theories this translates into [21, 22]

$$\eta_a R_{12}^a < 0. \quad (3.12)$$

For nonrelativistic scattering matrices in general and for the BDS S-matrix in particular it is less clear how to construct a simplified form of the physical state condition. Assuming that the pole of the magnon scattering matrix indeed describes a physical 2-magnon bound state, it appears possible to rephrase (3.11) on the rapidity plane u as the condition that the imaginary part of R_{12}^a has a definite sign. This can be simply stated if we notice that, even though the relevant S-matrices depend separately on the rapidities of the scattered excitations, the pole occurs for a fixed value for their difference. Then, the physical bound state condition (3.11) suggests that the imaginary part of the residues (taken with respect to the variable in which the pole occurs at a *positive* multiple of the imaginary unit) corresponding to physical bound states are positive

$$\text{Im}(R_{12}^a) > 0. \quad (3.13)$$

This is the condition we will use in the following. It is important to stress however that on the momentum plane the physical state condition appears to necessarily involve the phase η in a nontrivial way.

Solving the conditions (3.6) and (3.10) leads to the conclusion that the scattering matrix of a bound state against a fundamental excitation is given by:

$$S_{(12)3} = \sum_a (R_{12}^a)^{1/2} P_a (S_{13} S_{23}) \Big|_{\text{physical pole}} \sum_a (R_{12}^a)^{-1/2} P_a. \quad (3.14)$$

While derived here from weaker assumptions, the expression for the scattering matrix of a bound state off an elementary excitation is formally as in relativistically-invariant theories [21, 22].

The algorithm described here may be used to construct the scattering matrix of more complex bound states (i.e. bound states of more than one excitation) off elementary excitations as well as the scattering matrix of bound states against each other. For rank one S-matrices this expression simplifies considerably. In particular, there is no projection operator that is needed and moreover the residues of the poles cancel out and therefore are not needed (beyond making sure that the corresponding pole is physical).

Quite clearly, the fusion algorithm applies to a large class of S-matrices; in particular, it is not restricted to unit rank. However, unit rank sectors of larger S-matrices are particularly simple to analyze. As mentioned before, the spectrum of bound states obtained in such sectors naturally extends to representations of the manifest symmetry group of the original S-matrix. If the complete S-matrix is more symmetric, then these representations should in turn fit into representations of this larger symmetry group. It is in general unclear how this happens or whether the symmetry is linearly realized at the level of the bound states. In the following we will not address this issue and we will mostly restrict ourselves to unit rank sectors of the gauge and string theory S-matrix. We will however identify (in the small 't Hooft coupling limit) the S-matrix whose poles corresponds to the multiplet containing the 2-magnon bound state and construct the S-matrix of this multiplet off the “elementary” excitations.

4. BDS and AFS type S-matrices and the scattering of bound states

We will now apply the algorithm described in the previous section and determine the scattering matrix of multi-magnon bound states as encoded both in the BDS [1] and in the AFS [3] S-matrices. We will then take the strong coupling limit on the results obtained from the AFS S-matrix and compare it with the soliton scattering in the sigma model.

The part of the SU(2) sector S-matrix carrying information about the bound states is

$$S_{12}^{\text{BDS}} = \frac{u(p_1) - u(p_2) + i}{u(p_1) - u(p_2) - i} \quad u(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{1 + 4\bar{\lambda} \sin^2 \frac{p}{2}}. \quad (4.1)$$

Up to the precise form of $u(p)$, this S-matrix is the same as that describing magnon scattering in the Heisenberg chain. The different $u(p)$ has some nontrivial effects in that the actual physical parameter is the momentum rather than u and the bound states correspond to some complex values of p . It is nevertheless possible to figure out the real part of $u(p)$ corresponding to bound states: in particular, for a 2-magnon bound state it turns out [12, 14] that

$$u_2(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{4 + 4\bar{\lambda} \sin^2 \frac{p}{2}}. \quad (4.2)$$

A similar expression for a J -magnon bound state was obtained in [14] starting from the multi-magnon S-matrix (Bethe equations)

$$u_J(p) = \frac{1}{2} \cot \frac{p}{2} \sqrt{J^2 + 4\bar{\lambda} \sin^2 \frac{p}{2}}. \tag{4.3}$$

Here p denotes the total momentum of the bound state.

With this input it is easy to proceed with fusing (4.1) into the scattering of bound states.

4.1 BDS

We have used the pole in the BDS S-matrix (4.1) at $u(p_2) - u(p_1) = i$ as input in our simplified form of the physical state condition. It is not hard to see that $\text{Im}(\text{Res } S_{12}^{\text{BDS}}) = 2$. It follows then that the scattering matrix of the charge-2 bound state against an elementary magnon is

$$\begin{aligned} S_{J_2=2,1} &= \frac{u(p_2) - u_1 + i}{u(p_2) - u_1 - i} \frac{u(p_3) - u_1 + i}{u(p_3) - u_1 - i} \Big|_{\substack{u(p_2)=u_{J_2=2+i/2} \\ u(p_3)=u_{J_2=2-i/2}}} \\ &= \frac{u_{J_2=2} - u_1 + \frac{3i}{2}}{u_{J_2=2} - u_1 - \frac{3i}{2}} \frac{u_{J_2=2} - u_1 + \frac{i}{2}}{u_{J_2=2} - u_1 - \frac{i}{2}}. \end{aligned} \tag{4.4}$$

Out of the two poles the physical one is at $u_{J_2=2} - u_1 = \frac{3i}{2}$. The imaginary part of the residue of the other pole (in the sense described above) is

$$\text{Im} \left(\text{Res } S_{J_2=2,1} \Big|_{u_{J_2=2} - u_1 = \frac{i}{2}} \right) = -2 \tag{4.5}$$

implying that this second pole is unphysical.

Repeating J_2 times the fusion procedure for the physical pole it is quite easy to find that the scattering matrix of a J_2 -magnon bound state against an elementary magnon state is

$$S_{J_2,1} = \frac{u_{J_2} - u_1 + \frac{i}{2}(J_2 + 1)}{u_{J_2} - u_1 - \frac{i}{2}(J_2 + 1)} \frac{u_{J_2} - u_1 + \frac{i}{2}(J_2 - 1)}{u_{J_2} - u_1 - \frac{i}{2}(J_2 - 1)}. \tag{4.6}$$

The only physical pole of this S-matrix — which is the pole that should be used to raise J_2 by one unit — is at

$$u_{J_2} - u_1 = \frac{i}{2}(J_2 + 1). \tag{4.7}$$

Here, u_J continues to carry the interpretation of rapidity/spectral parameter but, as mentioned before, its expression in terms of the bound state momentum is different from that of elementary magnons and is given by (4.3).

We may further fuse the S-matrices (4.6) into those describing the scattering of a $J_2^{(1)}$ -magnon bound state against a $J_2^{(2)}$ -magnon bound state with $J_2^{(1)} > J_2^{(2)}$. The result is:

$$\begin{aligned}
 S_{J_2^{(1)}J_2^{(2)}}^{\text{BDS}} &= \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} - J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} - J_2^{(2)})} \times \\
 &\times \left[\prod_{l=1}^{J_2^{(2)}-1} \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} - J_2^{(2)} + 2l)}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} - J_2^{(2)} + 2l)} \right]^2 \times \\
 &\times \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} + J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} + J_2^{(2)})}. \tag{4.8}
 \end{aligned}$$

In this case both $u_{J_2^{(1)}}(p)$ and $u_{J_2^{(2)}}(p)$ are given by (4.3).

An interesting feature of (4.8) is that, besides simple poles, it also exhibits higher order poles. This structure was previously observed in integrable relativistic and nonrelativistic field theories [23]–[26], [7]. In relativistic theories they were associated to anomalous thresholds in the higher loop contributions to the scattering matrix of bound states. It is not currently clear whether the double-poles we find here have an interpretation from the standpoint of the nonrelativistic field theory whose resummed scattering matrix reproduces (4.1) or whether they can be reproduced only in the complete world sheet theory.

4.2 AFS

The results obtained above can be easily extended to include for the conjectured dressing phase connecting the gauge theory and the string theory S-matrices.

$$S_{\text{string}} = S_0 S_{\text{su}(2|2)^2} \rightarrow S_{\text{AFS}} = \sigma^2 S_{\text{BDS}}. \tag{4.9}$$

In uniform gauge the general structure of σ^2 is believed to have the form

$$\begin{aligned}
 \sigma^2(p_1, p_2) &\equiv e^{i\theta} \\
 &= \exp \left\{ i \sum_{rs} \left(\frac{\bar{\lambda}}{4} \right)^{\frac{1}{2}(r+s-1)} c_{rs}(\sqrt{\bar{\lambda}}) \left(q_r(p_1)q_s(p_2) - q_s(p_1)q_r(p_2) \right) \right\} \tag{4.10}
 \end{aligned}$$

where q_r are the higher conserved local charges. The expression of the coefficients $c_{rs}(\sqrt{\bar{\lambda}})$ is not known, but there exists indirect evidence that the first few terms in their expansion are ([3] and [4, 5], respectively)

$$c_{rs}(\sqrt{\bar{\lambda}}) = \delta_{s,r+1} + \frac{(-)^{r+s} - 1}{2\sqrt{\bar{\lambda}}} \frac{(r-1)(s-1)}{(r-1)^2 - (s-1)^2} + \dots \tag{4.11}$$

It is not hard to see using the resummed expression for (4.10) given in [2, 27] that the leading term in the dressing phase does not introduce additional poles in the 2-particle S-matrix besides those already present in S_{BDS} . It therefore follows that the contribution

of the residues of the physical pole to the bound state S matrix cancels out.⁶ Thus, the remaining contribution comes from multiplying together the dressing factors and evaluating them for the complex momenta describing the bound state formation. The result emulates the elementary magnon dressing phase (4.10) except that the higher local charges now are those of the corresponding bound states.

$$q_r^J(p_{\text{total}}) = \sum_{l=-J}^J q_r(a_l + ib_l). \quad (4.12)$$

This expression can be substantially simplified by recalling that $q_r(p)$ may be written in terms of the rapidities $u(p)$. In these variables the charges become

$$q_r^J = \frac{i}{r-1} \left[\frac{1}{x(u_J(p) + \frac{i}{2}J)^{r-1}} - \frac{1}{x(u_J(p) - \frac{i}{2}J)^{r-1}} \right]$$

with $x(u) = \frac{1}{2}(u + \sqrt{u^2 - \bar{\lambda}})$. (4.13)

Following our discussion in §3 as well as that in the previous subsection we will treat the rapidity u rather than $x(u)$ as fundamental variable. The appearance of $x(u)$ should be thought of as a shorthand for its expression in terms of the rapidity (or momentum).

With this clarifications, the conjectured general form of the string S -matrix (4.9) implies that the scattering matrix of a $J_2^{(1)}$ - and a $J_2^{(2)}$ -magnon bound state is:

$$S_{J_2^{(1)} J_2^{(2)}} = \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} + J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} + J_2^{(2)})} \times$$

$$\times \left[\prod_{l=1}^{J_2^{(2)}-1} \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} - J_2^{(2)} + 2l)}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} - J_2^{(2)} + 2l)} \right]^2 \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} - J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} - J_2^{(2)})} \times$$

$$\times \exp \left\{ i \sum_{rs} c_{rs}(\sqrt{\bar{\lambda}}) \left(\frac{\bar{\lambda}}{4} \right)^{\frac{1}{2}(r+s-1)} \begin{pmatrix} q_r^{J_2^{(1)}} & q_s^{J_2^{(2)}} \\ q_s^{J_2^{(1)}} & q_r^{J_2^{(2)}} \end{pmatrix} \right\}. \quad (4.14)$$

The self-similar property of the dressing phase is a direct consequence of its bilinear dependence on the higher local charges.

5. The large λ limit

For the purpose of comparison with the world sheet theory we must take the large 't Hooft coupling limit while keeping all momenta fixed. In this limit only the leading term in the coefficients $c_{rs}(\sqrt{\bar{\lambda}})$ is necessary. As emphasized in the introduction, there are in fact several distinct $\lambda \rightarrow \infty$ limits, which are distinguished by whether the length of the Bethe string is kept finite or is allowed to scale to infinity. Following our philosophy, we express

⁶This is valid in general if the incoming and outgoing particles belong to *the same* rank one sector.

the relevant higher local charges and the corresponding spectral parameters x in terms of the rapidities u and in terms of the bound state momenta:

$$\begin{aligned}
 x &= \frac{1}{2} \left(u + \sqrt{u^2 - \bar{\lambda}} \right) \tag{5.1} \\
 x^{\pm(J)}(p) &\equiv x(u_J(p) \pm \frac{i}{2}J) = \frac{e^{\pm \frac{i}{2}p}}{4 \sin \frac{p}{2}} \left(J + \sqrt{J^2 + 4\bar{\lambda} \sin^2 \frac{p}{2}} \right) \\
 q_r^J(p) &= \frac{i}{r-1} \left(\frac{1}{x^{+(J)}(p)^{r-1}} - \frac{1}{x^{-(J)}(p)^{r-1}} \right) \\
 &= \frac{2}{r-1} \sin \frac{1}{2}(r-1)p \left[\frac{\sqrt{J^2 + 4\bar{\lambda} \sin^2 \frac{p}{2}} - J}{\bar{\lambda} \sin \frac{p}{2}} \right]^{r-1}.
 \end{aligned}$$

These expressions hold for any nonvanishing J . At the same time, they define a prescription for taking the $\mathcal{J}_2^{(i)} \rightarrow 0$ limit or, alternatively, the large 't Hooft coupling limit. Indeed, from the standpoint of the spectral parameter of shifted argument $x(u_J \pm iJ/2)$, the limit $\mathcal{J} = J/\sqrt{\bar{\lambda}} \rightarrow 0$ makes $x^{\pm(J)}$ equal to leading order. However, when the limit is taken on their momentum space expressions (5.1) they remain different (cf. the second equation above, where the momentum dependent phase survives the limit $\mathcal{J} = J/\sqrt{\bar{\lambda}} \rightarrow 0$).

Let us first consider the case in which the length of the Bethe strings is kept fixed in the sigma model limit. It is then easy to see that all the local charges of the bound states become equal to those of the elementary magnons.⁷ Thus, in this limit the J -magnon bound states are indistinguishable from the elementary magnons, suggesting that they are distinguished only by quantum effects making suspicious a direct relation between these states and classical string solutions. A possible interpretation, which is the one we will adopt in the following, is that they should be considered as corresponding to the small charge limit of classical solutions. Proceeding along these lines, it is relatively easy to see that the sigma model limit of the fixed charge bound state scattering matrix is the same as that of “elementary” (i.e. unit charge) magnons

$$\theta_0 = \frac{\sqrt{\bar{\lambda}}}{2\pi} \left(\cos p_{J_2^{(2)}} - \cos p_{J_2^{(1)}} \right) \ln \frac{1 - \cos \frac{1}{2}(p_{J_2^{(1)}} - p_{J_2^{(2)}})}{1 - \cos \frac{1}{2}(p_{J_2^{(1)}} + p_{J_2^{(2)}})}, \tag{5.2}$$

where $p_{J_2^{(1)}}$ and $p_{J_2^{(2)}}$ are the momenta of the charge- $J_2^{(1)}$ and charge- $J_2^{(2)}$ magnons, respectively.

The other — more interesting — limit is when the lengths of the Bethe strings $J_2^{(i)}$ are scaled to infinity together with the 't Hooft coupling such that $\mathcal{J}_2^{(i)} = J_2^{(i)}/\sqrt{\bar{\lambda}}$ is kept fixed. This limit is analogous to that isolating the classical states in the SU(2) sector and, by analogy with the states with nonzero filling fraction, should be considered as corresponding to classical sigma model solutions.

⁷The same result for the energy is evident from [12]–[14]

$$E_J = \sqrt{J^2 + 4\bar{\lambda} \sin^2 \frac{1}{2}p} - J \xrightarrow{\bar{\lambda} \rightarrow \infty} E_J = 2\sqrt{\bar{\lambda}} |\sin \frac{1}{2}p|,$$

for any charge J .

Since the expressions for the higher local charges are different for the two scattered states, it is necessary to recompute the sum in the equation (4.14) while taking this into account. It is straightforward to do so by making use of the identity

$$\sum_{r \geq 2} \frac{1}{(r-1)r} \frac{\bar{\lambda}^{r-1/2}}{x^{r-1}y^r} = -\frac{1}{y\sqrt{\bar{\lambda}}} - \frac{xy/\bar{\lambda} - 1}{y/\sqrt{\bar{\lambda}}} \ln \left(1 - \frac{\bar{\lambda}}{xy} \right). \quad (5.3)$$

A small amount of algebra leads to⁸

$$\begin{aligned} \theta_0^{J_2^{(1)} J_2^{(2)}} = & 2(u_{J_2^{(2)}} - u_{J_2^{(1)}}) \ln \left[\frac{1 - \frac{\bar{\lambda}/4}{x^{+(J_2^{(1)})} x^{-(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{+(J_2^{(1)})} x^{+(J_2^{(2)})}}} \frac{1 - \frac{\bar{\lambda}/4}{x^{-(J_2^{(1)})} x^{+(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{-(J_2^{(1)})} x^{-(J_2^{(2)})}}} \right] \\ & + i(J_2^{(1)} + J_2^{(2)}) \ln \left[\frac{1 - \frac{\bar{\lambda}/4}{x^{-(J_2^{(1)})} x^{+(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{+(J_2^{(1)})} x^{-(J_2^{(2)})}}} \right] + i(J_2^{(1)} - J_2^{(2)}) \ln \left[\frac{1 - \frac{\bar{\lambda}/4}{x^{+(J_2^{(1)})} x^{+(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{-(J_2^{(1)})} x^{-(J_2^{(2)})}}} \right]. \end{aligned} \quad (5.4)$$

Clearly, $\theta_0^{J_2^{(1)} J_2^{(2)}}$ reduces to θ_{AFS} of [27, 2] for $J_2^{(1)} = J_2^{(2)} = 1$, as it should. It is also trivial to recover the phase (5.2). Indeed, by keeping both $J_2^{(1)}$ and $J_2^{(2)}$ fixed while taking the large λ limit it is easy to see that the second line is subleading while the first line immediately leads to (5.2).

If either $J_2^{(1)}$ or $J_2^{(2)}$ or both scale as $\sqrt{\lambda}$ the second line in equation (5.4) is no longer subleading, being of the same order as the first line. It is in fact possible to scale away the 't Hooft coupling and express the scattering matrix in terms of the ratios $\mathcal{J}^{(i)} = J^{(i)}/\sqrt{\lambda}$.

An interesting observation is that, if $J_2^{(2)}/\sqrt{\lambda}$ is fixed in the large $\sqrt{\lambda}$ limit, there appears to be an additional *leading order* contribution to the magnon scattering phase: there are $J_2^{(2)}$ factors in $S_{J_2^{(1)} J_2^{(2)}}^{\text{BDS}}$ and thus its logarithm scales to infinity as $\sqrt{\lambda}$. The potential shift of $\theta_0^{J_2^{(1)} J_2^{(2)}}$ is

$$\begin{aligned} \delta\theta_0^{J_2^{(1)} J_2^{(2)}} = & i \ln \left\{ \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} - J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} - J_2^{(2)})} \times \right. \\ & \times \left[\prod_{l=1}^{J_2^{(2)}-1} \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} - J_2^{(2)} + 2l)}{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} - J_2^{(2)} + 2l)} \right]^2 \times \\ & \left. \times \frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} + J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} + J_2^{(2)})} \right\}. \end{aligned} \quad (5.5)$$

There are two possible standpoints regarding the interpretation of this correction to the dressing phase contribution to the scattering phase of magnon bound states:

1. $\delta\theta_0^{J_2^{(1)} J_2^{(2)}}$ contains $J_2^{(2)}$ terms and thus, under our assumptions, is of the same order as the contribution of the dressing phase. It is therefore tempting to expect that it is also visible in the classical sigma model.

⁸All the sums can be performed using the momentum representation of the higher local charges (5.1).

2. $\delta\theta_0^{J_2^{(1)}J_2^{(2)}}$ is not proportional to $\sqrt{\lambda}$ (though scales like it under our assumptions) and thus it should not be included in a comparison with semiclassical string calculations which — by construction — yield contributions proportional to $\sqrt{\lambda}$ to all quantities. From this standpoint the $\sqrt{\lambda}$ dependence of (5.5) suggests that $\delta\theta_0^{J_2^{(1)}J_2^{(2)}}$ has — from the sigma model perspective — a 1-loop origin and the apparent semiclassical scaling is due to a “coherent superposition” of quantum effects.

In §7 we will see that both standpoints have an interpretation in the world sheet sigma model. To analyze this issue we will need a more tractable form for $\delta\theta_0^{J_2^{(1)}J_2^{(2)}}$. Up to corrections of the order of $1/J_2^{(i)}$ this can be done by replacing the sum of logarithms by their integral.⁹ The result is:

$$\begin{aligned} \delta\theta_0^{J_2^{(1)}J_2^{(2)}} = & 2(u_{J_2^{(2)}} - u_{J_2^{(1)}}) \ln \left[\frac{(u_{J_2^{(1)}} - u_{J_2^{(2)}})^2 + \frac{1}{4}(J_2^{(1)} - J_2^{(2)})^2}{(u_{J_2^{(1)}} - u_{J_2^{(2)}})^2 + \frac{1}{4}(J_2^{(1)} + J_2^{(2)})^2} \right] \\ & + i(J_2^{(1)} + J_2^{(2)}) \ln \left[\frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} + J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} + J_2^{(2)})} \right] \\ & - i(J_2^{(1)} - J_2^{(2)}) \ln \left[\frac{u_{J_2^{(1)}} - u_{J_2^{(2)}} + \frac{i}{2}(J_2^{(1)} - J_2^{(2)})}{u_{J_2^{(1)}} - u_{J_2^{(2)}} - \frac{i}{2}(J_2^{(1)} - J_2^{(2)})} \right]. \end{aligned} \tag{5.6}$$

It is worth mentioning that an intermediate regime between the two limits discussed above consists of keeping one charge fixed while scaling the other one to infinity together with λ . Since in our use of the fusion rules we assumed that $J_2^{(1)} > J_2^{(2)}$, it is natural that we keep $J_2^{(2)}$ fixed. We will not consider separately this regime, as it may be trivially obtained as the $J_2^{(2)}/\sqrt{\lambda} \rightarrow 0$ of the previous regime. Similarly, if we take the limit $J_2^{(i)}/\sqrt{\lambda} \rightarrow 0$ for both $i = 1$ and $i = 2$ we recover the first limit (5.2). Assuming that the general limit agrees with the sigma model scattering, this observation justifies the interpretation of the fixed charge solitons as the small charge limit of solitons present in the SU(2) sector. This is similar to interpreting BMN states as small charge limits of classical solutions of the sigma model.

5.1 On other sectors

Let us briefly comment on the other rank one sectors — SU(1|1) and SL(2) — as well as on the full S-matrix. Using the fact that the dressing phase does not — at least to this order — contain poles describing propagating modes, it suffices to analyze

$$S_\eta = \left(\frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \right)^\eta \frac{1 - \frac{\bar{\lambda}}{4x_1^+ x_2^-}}{1 - \frac{\lambda}{4x_1^- x_2^+}} \tag{5.7}$$

for $\eta = 0$ and $\eta = -1$.

⁹The calculation is formally identical had we chosen to express it as a large λ limit. This however obscures the fact that the reason this replacement is reliable is that the charges $J_2^{(i)}$ are large.

It is relatively easy to search for poles in these scattering matrices on the rapidity plane u . The result is that for $\eta = 0$ there are no poles while for $\eta = -1$ there is a pole which does not satisfy the physical state condition. Thus, there is no bound state of two magnons in the $SU(1|1)$ and the $SL(2)$ sectors. This latter observation matches the conclusion of [14] that the only bound states in the $SL(2)$ sector contain infinitely many magnons.

Naively these observations present a puzzle. While symmetry considerations suggest that there should exist bound states in other sectors, the absence of bound states in other unit rank sectors may suggest that the bound states found in the $SU(2)$ sector do not have a supersymmetric completion. This is however not true and the “missing” bound states may be identified as poles in the $SU(2|2)^2$ -invariant S-matrix [28] rather than the magnon S-matrix. Perhaps the easiest way to see this is to take the small coupling limit where the $SU(2|2)$ symmetry becomes manifest

$$S_{SU(2|2)} = \frac{1}{2} E_{12} S_0 \left[(1 - \mathcal{P}_{12}) + \frac{u_2 - u_1 + i}{u_2 - u_1 - i} (1 + \mathcal{P}_{12}) \right] \quad (5.8)$$

where E_{12} is the operator interchanging the rapidities of the excitations introduced in equation (3.8) (or, alternatively, switches the labels of the two excitations while leaving the momenta unchanged). Also, \mathcal{P} is the graded permutation operator and thus $(1 \pm \mathcal{P}_{12})$ are projectors onto the graded symmetric and graded antisymmetric representations.

Thus, at vanishing 't Hooft coupling, $S_{SU(2|2)}$ contain a physical pole at $u_2 - u_1 = i$ whose residue is proportional to the projector onto the two-index graded symmetric representation of $SU(2|2)$. The vanishing 't Hooft coupling limit of the $\mathcal{N} = 4$ S-matrix [28]

$$S_{\mathcal{N}=4} = S_{SU(2|2)} \otimes S_{SU(2|2)} \frac{u_2 - u_1 - i}{u_2 - u_1 + i} \quad (5.9)$$

inherits this pole. Its corresponding residue is

$$\text{Res } S_{\mathcal{N}=4} \Big|_{u_2 - u_1 = i} = 2i (1 + \mathcal{P}_{12}) \otimes (1 + \mathcal{P}_{12}) \quad (5.10)$$

and projects onto the tensor product of two 2-index graded symmetric representation of $SU(2|2)$. One of the states in this representation is the bound state visible in the magnon S-matrix.

Thus, at vanishing 't Hooft coupling, the $SU(2|2)^2$ -symmetric S-matrix contains sufficiently many poles to realize a representations of the $\mathcal{N} = 4$ symmetry algebra and the apparent contradiction outlined in the beginning of this section is resolved in this limit. While this analysis obviously does not directly apply at the level of the sigma model, it is nevertheless interesting to note that it does not follow the standard pattern of bosonic sigma models. There the (CDD) dressing factor implies that bound states occur in antisymmetric representations of the symmetry group and one may have expected here graded-antisymmetric representations to occur.

Using the fusion algorithm it is easy to construct the scattering matrix of bound states and elementary excitations. The calculation is simplified by the fact that the scattering of two $SU(2|2)$ excitations produces a single physical bound state and thus, as for unit rank

sectors, the precise value of the residue is not relevant.

$$S_{1b} = E_{b1} S_0(u_b + i/2, u_1) S_0(u_b - i/2, u_1) \times \left[P_{\{2,1,0,\dots\}} + \frac{u_b - u_1 + \frac{3i}{2}}{u_b - u_1 - \frac{3i}{2}} \frac{u_b - u_1 + \frac{i}{2}}{u_b - u_1 - \frac{i}{2}} P_{\{3,0,0,\dots\}} \right] \quad (5.11)$$

where $P_{\{j_1, j_2, j_3, \dots\}}$ denotes the projector onto the representation whose super-Young tableau has j_k boxes on the k -th row and u_b denotes the rapidity of the 2-particle bound state (4.2). Similarly to the representation of the 2-particle bound state, the structure of the S-matrix above is again different from that in the usual bosonic sigma models. Similarly to the SU(2)-magnon, the S-matrix (5.11) has one unphysical and one physical pole, the latter of which may be used to construct further bound states and their corresponding scattering matrices.

The situation is less clear at $\lambda \neq 0$, but it appears unlikely that the representation of the bound state changes as a function of the coupling constant.¹⁰ The main difficulty with performing the analysis above for the finite- λ S-matrix of [28] comes from the fact that the symmetry algebra is no longer manifestly realized. It is however trivial to see that the S-matrix continues — for a finite range of λ — to have physical poles at the expected position $u_2 - u_1 = i$. While the residue is no longer a projector onto the 2-index graded-symmetric representation of SU(2|2), this symmetry is inherited from the full S-matrix and this continues to be realized (though not manifestly).

6. Scattering of higher charge world sheet solitons

The solitons corresponding to large charge bound states have at least two representations. They can be explicitly written as classical solutions of the world sheet sigma model in which one relaxes the level-matching condition [11], [14]–[16]. Classically, the sigma model can be mapped into the complex sine-Gordon theory [13]. The solitons of this theory carry nonvanishing charge with respect to the U(1) symmetry of this model and have been argued to correspond to multi-magnon bound states. There is a one-to-one relation between the complex sine-Gordon U(1) charge and the number of bound magnons.

We will first recall the results of [18, 19] and then translate them to the variables corresponding to the world sheet sigma model and compare the result with the large 't Hooft coupling limit of the scattering matrices derived in the previous section.

¹⁰It is notable however that the imaginary parts of the residues of some of the elements of the SU(2|2)-invariant S-matrix change sign at rapidity-dependent values of the 't Hooft coupling. This should not be interpreted as a change in the spectrum of the theory. Inspecting the relevant matrix elements leads to the conclusion that the poles with indefinite-sign residue arise in elements which vanish in the $\lambda \rightarrow 0$ limit while the poles with a negative imaginary part for their residues combine in this limit into the projector onto the graded symmetric representation of SU(2|2). This suggests that, similarly to non-simply-laced affine Toda theories [9, 10], the poles with indefinite-sign residue — though simple — should not be interpreted as corresponding to bound states. I would like to thank Patrick Dorey for pointing out this possibility.

6.1 Classical scattering of solitons

To find the classical scattering phase of solitons it is necessary to construct the solution describing their scattering, extract the time delay accumulated in the scattering process and then integrate it. The construction of arbitrary-charge scattering solutions directly in the world sheet sigma model is quite complicated [15].

Classically, the S^3 sigma model can be mapped to the Complex sine-Gordon theory. In particular, there is a one to one correspondence between the scattering solutions of CsG solitons and those of the sigma model. This enables us to use existing results on their classical scattering phase; we only need to carefully translate to the dispersion relation of the sigma model.

Similarly to the scattering solutions of the sigma model, the analytic expressions for the solutions of CsG are extremely cumbersome, but are however known for arbitrary charges (see e.g. [18] for details). The time delay was extracted in [18, 19] from the large-time asymptotics of these expressions. The CsG solitons are characterized by their rapidities β and by an additional quantity α related to their U(1) charge. Let us consider two solitons with rapidities β_1, β_2 and U(1) parameters α_1 and α_2 . Then, the time delay due to the collision is [19]:

$$\Delta t(\beta_1, \beta_2, \alpha_1, \alpha_2) = \frac{2}{\sinh(\beta_1) \cos(\alpha_1)} \log \left| \frac{\sinh\left[\frac{1}{2}(\beta_1 - \beta_2) + \frac{i}{2}(\alpha_1 - \alpha_2)\right]}{\cosh\left[\frac{1}{2}(\beta_1 - \beta_2) + \frac{i}{2}(\alpha_1 + \alpha_2)\right]} \right|. \quad (6.1)$$

The semi-classical phase shifts $\delta(E_1, E_2)$ for soliton-soliton scattering is then determined by the WKB formula [29]

$$\frac{d\delta(E_1, E_2)}{dE_1} = \Delta t(E_1, E_2, J_2^{(1)}, J_2^{(2)}) \longrightarrow \delta(E_1, E_2) = \int dE_1 \Delta t(E_1, E_2, J_2^{(1)}, J_2^{(2)}). \quad (6.2)$$

We need to transform this in variables appropriate to the sigma model, i.e. we need to use the bound state dispersion relation rather than the relativistic dispersion relation of the CsG theory. First, (6.1) can be recast into

$$\Delta t(\beta_1, \beta_2, \alpha_1, \alpha_2) = \frac{2}{\sinh(\beta_1) \cos(\alpha_1)} \ln \frac{\cosh(\beta_1 - \beta_2) - \cos(\alpha_1 - \alpha_2)}{\cosh(\beta_1 - \beta_2) + \cos(\alpha_1 + \alpha_2)}. \quad (6.3)$$

The goal is then to express α_i and β_i in terms of the charges $J_2^{(i)}$ and the momenta p_i .

As discussed in [13], the precise soliton dispersion relation as well as the expressions of the sigma model momenta and angular momenta in terms of the CsG parameters are

$$J_2^{(i)} = 2\sqrt{\lambda} \tan \alpha_i \sin^2 \frac{1}{2} p_i, \quad E_i = \sqrt{J_2^{(i)2} + 4\lambda \sin^2 \frac{1}{2} p_i}, \quad \sinh \beta_i = \frac{\cos \alpha_i}{\tan \frac{1}{2} p_i}. \quad (6.4)$$

Since the gauge theory quantities depend only on the square of the charges, there is a potential sign ambiguity in the first equation above. For multi-soliton configurations it applies for each soliton independently.¹¹ Alternatively, we may choose to fix this ambiguity as in (6.4) and interpret solitons with negative charge as anti-solitons.

¹¹The charges and energies of each soliton are identified from the configurations in which the solitons are widely separated.

Using (6.4) implies, after a small amount of algebra, that the relevant combinations of rapidities and α -parameters are

$$\begin{aligned} \cos(\alpha_1 \mp \alpha_2) &= \frac{4\bar{\lambda} \sin^2 \frac{1}{2}p_1 \sin^2 \frac{1}{2}p_2 \pm J_2^{(1)} J_2^{(2)}}{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^4 \frac{1}{2}p_1} \sqrt{J_2^{(2)2} + 4\bar{\lambda} \sin^4 \frac{1}{2}p_2}} \\ \cosh(\beta_1 - \beta_2) &= \frac{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^2 \frac{1}{2}p_1} \sqrt{J_2^{(2)2} + 4\bar{\lambda} \sin^2 \frac{1}{2}p_2} - \bar{\lambda} \sin p_1 \sin p_2}{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^4 \frac{1}{2}p_1} \sqrt{J_2^{(2)2} + 4\bar{\lambda} \sin^4 \frac{1}{2}p_2}}. \end{aligned} \quad (6.5)$$

Thus, in terms of the sigma model variables, the time delay becomes

$$\begin{aligned} \Delta t &= \frac{J_2^{(1)2} + 4\bar{\lambda} \sin^4 \frac{1}{2}p_1}{4\bar{\lambda} \sin^3 \frac{1}{2}p_1 \cos \frac{1}{2}p_1} \times \\ &\times \log \frac{E_1(p_1, J_2^{(1)}) E_2(p_2, J_2^{(2)}) - \bar{\lambda} \sin p_1 \sin p_2 - (4\bar{\lambda} \sin^2 \frac{1}{2}p_1 \sin^2 \frac{1}{2}p_2 + J_2^{(1)} J_2^{(2)})}{E_1(p_1, J_2^{(1)}) E_2(p_2, J_2^{(2)}) - \bar{\lambda} \sin p_1 \sin p_2 + (4\bar{\lambda} \sin^2 \frac{1}{2}p_1 \sin^2 \frac{1}{2}p_2 - J_2^{(1)} J_2^{(2)})}. \end{aligned} \quad (6.6)$$

Last, using trivially the dispersion relation (6.4), the integration measure in (6.2) becomes

$$dE_1 = dp_1 \frac{dE_1}{dp_1} = dp_1 \frac{2\bar{\lambda} \sin \frac{1}{2}p_1 \cos \frac{1}{2}p_1}{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^2 \frac{1}{2}p_1}} \quad (6.7)$$

With these ingredients we may proceed to compare the classical soliton-soliton scattering with the predictions of the fused AFS S-matrix. It is worth mentioning that the scattering phase constructed from (6.6) and (6.7) is invariant under the simultaneous transformation $(J_2^{(1)}, J_2^{(2)}) \rightarrow (-J_2^{(1)}, -J_2^{(2)})$, but changes nontrivially under $(J_2^{(1)}, J_2^{(2)}) \rightarrow (-J_2^{(1)}, J_2^{(2)})$.

7. Comparison

Let us now consider separately the two regimes we introduced before: fixed $J_2^{(i)}$ and fixed $J_2^{(i)}/\sqrt{\lambda}$ as $\lambda \rightarrow \infty$. Following the philosophy described in the introduction, we may obtain the former from the latter as the limit of vanishing $J_2^{(i)}/\sqrt{\lambda}$. In this sense the fixed charge magnon bound state may be interpreted as a classical solution of the sigma model.

For finite $J_2^{(i)}/\sqrt{\lambda}$ we will discuss separately the dressing phase and complete AFS S-matrix; as mentioned in §4, in both cases we will find a sigma model interpretation.

7.1 Large $\bar{\lambda}$ fixed $J_2^{(i)}$

It is quite trivial to see that in this limit (with fixed p_i) all $J_2^{(i)}$ -dependence becomes irrelevant. Indeed, the expression for the time delay Δt can easily be written in terms of $J_2^{(i)}/\sqrt{\lambda}$, which vanishes in this limit. Thus, the time delay reduces to that of [11], implying that the semi-classical phase shift describing the scattering of fixed charge magnon bound

states is

$$\begin{aligned} \delta_{\text{ws}} &= \sqrt{\bar{\lambda}} \int dp_1 \sin \frac{1}{2} p_1 \log \left[\frac{1 - \cos \frac{1}{2} p_1 \cos \frac{1}{2} p_2 - \sin \frac{1}{2} p_1 \sin \frac{1}{2} p_2}{1 - \cos \frac{1}{2} p_1 \cos \frac{1}{2} p_2 + \sin \frac{1}{2} p_1 \sin \frac{1}{2} p_2} \right] \\ &= 2\sqrt{\bar{\lambda}} \left[\left(\cos \frac{1}{2} p_2 - \cos \frac{1}{2} p_1 \right) \ln \frac{1 - \cos \frac{1}{2} (p_1 - p_2)}{1 - \cos \frac{1}{2} (p_1 + p_2)} - p_1 \sin \frac{1}{2} p_2 \right]. \end{aligned} \quad (7.1)$$

As in [11], the apparent difference between the prediction of the fusion construction and that of CsG theory may be ascribed to a change of gauge from the perhaps more standard uniform gauge – if we were to write Bethe equations for states constructed out of magnon bound states.¹² The difference cancels (trivially) between the left- and the right-hand sides of the Bethe equations. This is because the difference between the string length as inherited from the CsG analysis and that in the uniform gauge is the energy of the solution which can further be written as the sum over the various (giant) magnons.

7.2 The general case

The analysis of the general limit

$$\bar{\lambda} \rightarrow \infty \quad \text{with} \quad \mathcal{J}_2^{(i)} = J_2^{(i)} / \sqrt{\bar{\lambda}} = \text{fixed} \quad (7.2)$$

is somewhat more complicated. The goal is to test whether the soliton scattering phase constructed in the previous section

$$\begin{aligned} \delta_{\text{ws}}^{J_2^{(1)}, J_2^{(2)}} &= \frac{1}{2} \int \frac{dp_1}{\sin^2 \frac{1}{2} p_1} \frac{J_2^{(1)2} + 4\bar{\lambda} \sin^4 \frac{1}{2} p_1}{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_1}} \times \\ &\times \log \frac{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_1} \sqrt{J_2^{(2)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_2} - 4\bar{\lambda} \sin \frac{1}{2} p_1 \sin \frac{1}{2} p_2 \cos \frac{1}{2} (p_1 - p_2) - J_2^{(1)} J_2^{(2)}}{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_1} \sqrt{J_2^{(2)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_2} - 4\bar{\lambda} \sin \frac{1}{2} p_1 \sin \frac{1}{2} p_2 \cos \frac{1}{2} (p_1 + p_2) - J_2^{(1)} J_2^{(2)}} \end{aligned} \quad (7.3)$$

is related to equation (5.4).

It turns out that it is possible to directly compute the integral above. To this end we first notice that the argument of the logarithm in (7.3) may be written as

$$\begin{aligned} &\frac{E_1 E_2 - J_2^{(1)} J_2^{(2)} - 4\bar{\lambda} \sin \frac{1}{2} p_1 \sin \frac{1}{2} p_2 \cos \frac{1}{2} (p_1 - p_2)}{E_1 E_2 - J_2^{(1)} J_2^{(2)} - 4\bar{\lambda} \sin \frac{1}{2} p_1 \sin \frac{1}{2} p_2 \cos \frac{1}{2} (p_1 + p_2)} = \\ &= \frac{1 - \frac{\bar{\lambda}/4}{x^{+(-J_2^{(1)})} x^{-(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{+(-J_2^{(1)})} x^{+(J_2^{(2)})}}} \frac{1 - \frac{\bar{\lambda}/4}{x^{-(-J_2^{(1)})} x^{+(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{-(-J_2^{(1)})} x^{-(J_2^{(2)})}}}. \end{aligned} \quad (7.4)$$

This identity may be established either analytically or numerically. The definition of $x^{\pm(-J)}$ is given by the second equal sign in the second equation (5.1) with the replacement $J \rightarrow -J$:

$$x^{\pm(-J)}(p) = \frac{e^{\pm \frac{1}{2} p}}{4 \sin \frac{p}{2}} \left(-J + \sqrt{J^2 + 4\bar{\lambda} \sin^2 \frac{p}{2}} \right). \quad (7.5)$$

¹²It is worth emphasizing that Bethe equations constructed out of the bound state scattering matrix describe fewer states than the Bethe equations for elementary magnons.

Furthermore, we also notice that the coefficient of the logarithm under the integral sign in equation (7.3) is a simple total derivative:

$$2 \frac{d}{dp_1} (u_{J_2^{(2)}} - u_{J_2^{(1)}}) = \frac{1}{2 \sin^2 \frac{1}{2} p_1} \frac{J_2^{(1)2} + 4\bar{\lambda} \sin^4 \frac{1}{2} p_1}{\sqrt{J_2^{(1)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_1}} \quad (7.6)$$

Thus, integrating by parts we immediately obtain the first line in (5.4) with the transformation $J_2^{(1)} \rightarrow -J_2^{(1)}$.

The remaining integral can also be evaluated, though with somewhat more effort. The final answer (which may be trivially checked by differentiating with respect to p_1 and then comparing numerically with the integrand in (7.3)) turns out to be

$$\delta_{\text{ws}}^{-J_2^{(1)}, J_2^{(2)}} = \delta_{\text{ws}}^{J_2^{(1)}, -J_2^{(2)}} = \theta_0^{J_2^{(1)} J_2^{(2)}} - p_1 \left(\sqrt{J_2^{(2)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_2} - J_2^{(2)} \right). \quad (7.7)$$

This result may be interpreted in two different ways.

1. If we choose to make use of the sign ambiguity in the relation between the J_2 and the CsG parameter α (see below equation (6.4)), then we may freely change the sign of one of the charges on the left hand side of the equation above. Thus, with this prescription, the sigma model scattering of charge- $J_2^{(1)}$ and charge- $J_2^{(2)}$ solitons appears to reproduce the semiclassical dressing phase.
2. If we choose to relate both $J_2^{(1)}$ and $J_2^{(2)}$ and the corresponding CsG parameters α_i as in the equation (6.4), then the sigma model soliton-*antisoliton* scattering reproduces the semiclassical dressing phase.

In both instances, the difference between the left and right-hand-sides is the natural extension to finite charge states of the analogous term in (7.1) and as in that case goes away at the level of the Bethe equations.¹³

The BDS-like part of the gauge theory bound state scattering matrix may be included in this comparison by making use of further identities. Indeed, it is not hard to show that

$$\frac{1 - \frac{\bar{\lambda}/4}{x^{+(-J_2^{(1)})} x^{- (J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{+(-J_2^{(1)})} x^{+(J_2^{(2)})}}} \frac{1 - \frac{\bar{\lambda}/4}{x^{-(-J_2^{(1)})} x^{+(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{-(-J_2^{(1)})} x^{- (J_2^{(2)})}}} = \frac{1 - \frac{\bar{\lambda}/4}{x^{+(J_2^{(1)})} x^{- (J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{+(J_2^{(1)})} x^{+(J_2^{(2)})}}} \frac{1 - \frac{\bar{\lambda}/4}{x^{- (J_2^{(1)})} x^{+(J_2^{(2)})}}}{1 - \frac{\bar{\lambda}/4}{x^{- (J_2^{(1)})} x^{- (J_2^{(2)})}}} \times \frac{(u_{J_2^{(1)}} - u_{J_2^{(2)}})^2 + \frac{1}{4} (J_2^{(1)} - J_2^{(2)})^2}{(u_{J_2^{(1)}} - u_{J_2^{(2)}})^2 + \frac{1}{4} (J_2^{(1)} + J_2^{(2)})^2}. \quad (7.8)$$

This implies that, with the relation between both $J_2^{(1)}$ and $J_2^{(2)}$ and the corresponding CsG parameters α_i as in equation (6.4), the sum of the first lines of equations (5.4) and (5.5) reproduces the result of the integration by parts in equation (7.3).

¹³This emphasizes that, since S-matrices are gauge-dependent, two of them cannot be directly compared unless they are computed in the same gauge. Rather, one should compare physical quantities. Some of the gauge dependence disappears at the level of the Bethe equations, which is what is used in [11] and here.

As before, it is possible (with some effort) to perform the remaining integral with the result

$$\delta_{\text{ws}}^{J_2^{(1)}, J_2^{(2)}} = \theta_0^{J_2^{(1)}, J_2^{(2)}} + \delta\theta_0^{J_2^{(1)}, J_2^{(2)}} - p_1 \left(\sqrt{J_2^{(2)2} + 4\bar{\lambda} \sin^2 \frac{1}{2} p_2} - J_2^{(2)} \right). \quad (7.9)$$

Therefore, up to the same term accounting for the difference of gauge choice, the sigma model scattering phase reproduces both the dressing phase as well as contribution of the fused BDS S-matrix.

8. The small momentum limit and further checks

The two results of the previous section appear somewhat surprising: both the complete AFS scattering matrix and the dressing phase have independent world sheet interpretation. They are the soliton-soliton and soliton-antisoliton scattering phases. We will now provide independent evidence that this is indeed correct.

To this end we will consider the limit in which the world sheet soliton become regular string states:

$$J_2^{(i)} \rightarrow 1 \quad p \rightarrow 0 \quad \bar{\lambda} p^2 = \text{fixed}. \quad (8.1)$$

In this limit we should be able to compare (up to gauge and coordinate artifacts) the scattering phases (7.7) and (7.9) with the scattering amplitudes of the world sheet fields corresponding to gauge theory magnons.

It is relatively clear that $\delta_{\text{ws}}^{J_2^{(1)}, J_2^{(1)}}$ becomes, in this limit, the scattering of the holomorphic world sheet scalars of the Landau-Lifshitz model [30, 31]. A potential subtlety relates to the fate of $\delta_{\text{ws}}^{-J_2^{(1)}, J_2^{(1)}}$ in that it should be related to the scattering fields of opposite R-charge. Due to its nonrelativistic nature, the world sheet image of the gauge theory SU(2) sector — the Landau-Lifshitz model — capture only the scattering of likewise charge fields. Thus, the scattering amplitude which should match $\delta_{\text{ws}}^{-1,1}$ lies outside the SU(2) sector.¹⁴ Consequently, to identify on the world sheet the low momentum and low angular momentum limit of dyonic giant magnon scattering we should analyze the sigma model without restricting the fields to their positive energy modes. Relaxing this restriction allows fields of opposite R-charges in the initial state. In the full string theory the scattering of neutral configurations (such as solitons and antisolitons) leads in the final state to (essentially) all possible neutral combination of fields. Thus, testing e.g. the factorization of the S-matrix necessarily requires (some of) the other fields of the theory. The soliton-antisoliton scattering phase obtained from the CsG theory captures *only* the “exclusive” amplitude with the same final and initial states.

The dyonic giant magnons have, by construction, an infinitely large angular momentum J_1 . It appears therefore natural to use, similarly to [32], a gauge fixing it. The complete semiclassical world sheet scattering matrix in this gauge will be discussed elsewhere [33].

¹⁴This conclusion should in fact not be unexpected. Up to gauge and coordinate artifacts the dressing phase is universal and thus its world sheet interpretation needs not be restricted to a single sector.

The action restricted to $R \times S^3$ is easy to construct following the strategy of [34]. Starting with the $R \times S^3$ metric

$$ds^2 = -dt^2 + \frac{(1 - y\bar{y})^2}{(1 + y\bar{y})^2} d\phi^2 + \frac{4dyd\bar{y}}{(1 + y\bar{y})^2}, \quad (8.2)$$

performing 2d duality along ϕ and fixing the uniform gauge $t = \tau$ and $\tilde{\phi} = \frac{J_1}{\sqrt{\lambda}}\sigma$, redefining the spatial coordinate $\sigma = \frac{2\pi}{J_1}x$ and then scaling J_1 to infinity we obtain an action defined on the plane ($S = \int d\tau \int_0^\infty dx L$) and the lagrangian

$$L = \frac{1}{2} [(\partial_\tau y)^2 - \bar{\lambda}(\partial_x y)^2 - y^2] + \frac{1}{2} (\bar{\lambda}(\partial_x y)^2 - (\partial_\tau y)^2) (\bar{\lambda}(\partial_x \bar{y})^2 - (\partial_\tau \bar{y})^2) + 2\bar{\lambda} y\bar{y} \partial_x y \partial_x \bar{y} - \frac{1}{2} y^2 \bar{y}^2 + \dots, \quad (8.3)$$

where the ellipsis stands for terms with more than six fields. With the mode expansion

$$y = \int \frac{dk_1}{\sqrt{2k_0}} (a(k)e^{-ik \cdot z} + b(k)^\dagger e^{+ik \cdot z}), \quad k_0 \equiv e(k_1) = \sqrt{1 + \bar{\lambda}k_1^2} \quad (8.4)$$

which leads to canonical commutation relations, the amplitude of the process $yy \rightarrow yy$ is [33]

$$S_{y(p_1)y(p_2) \rightarrow y(p_1)y(p_2)} = \frac{1}{4} \frac{1}{p_1 e(p_2) - p_2 e(p_1)} \left[2 (\bar{\lambda} p_1 p_2 - e(p_1) e(p_2))^2 + 2\bar{\lambda} (p_1 + p_2)^2 - 2 \right] \\ = -(p_1 - p_2) + (\theta_0^{11} + \delta\theta_0^{11}) \Big|_{\substack{p_i \rightarrow 0 \\ \bar{\lambda} p_i^2 = \text{fixed}}} \quad (8.5)$$

is the same (up to the change of gauge) as in [31]. Thus, in the small momentum limit, the soliton-soliton scattering phase becomes the scattering phase of the world sheet fields corresponding to the “elementary” magnons.

The amplitude of the process $y(p_1)\bar{y}(p_2) \rightarrow y(p_1)\bar{y}(p_2)$ can be equally well extracted from (8.3). It is [33]

$$S_{y(p_1)\bar{y}(p_2) \rightarrow y(p_1)\bar{y}(p_2)} = \frac{1}{4} \frac{1}{p_1 e(p_2) - p_2 e(p_1)} \left[2 (\bar{\lambda} p_1 p_2 - e(p_1) e(p_2))^2 + 2\bar{\lambda} (p_1 - p_2)^2 - 2 \right] \\ = -(p_1 - p_2) + \theta_0^{11} \Big|_{\substack{p_i \rightarrow 0 \\ \bar{\lambda} p_i^2 = \text{fixed}}} \quad (8.6)$$

which, up to the gauge and coordinate artifacts leading to the first term above and the last term in (7.7), agrees with our expectations and confirms the identification of the sigma model $(-J_2^{(1)}, J_2^{(2)})$ soliton-antisoliton scattering phase with the fused dressing phase.

9. Summary and discussions

In an integrable quantum field theory the scattering of the bound states is determined by the S-matrix of its constituents. If independent calculations are available for some of its limits, the comparison tests the consistency of the scattering matrix of the constituents of the bound state.

We have re-analyzed the bootstrap construction without assuming Lorentz-invariance and found the conditions that a simple pole of the scattering matrix describes a physical bound state. Formally it is unchanged from the equivalent condition in a Lorentz-invariant theory. Using this condition we have constructed the scattering matrix of bound states of arbitrary (fixed or/and scaling) J_2 charge under the assumption that their constituents scatter with the BDS and AFS-type S-matrices. The analytic structure of the result is somewhat richer than that of the “elementary” magnons, exhibiting both simple and double-poles.

We compared the result based on the AFS-type magnon S-matrix with the scattering of the corresponding world sheet solitons. Their connection to the complex sine-Gordon theory provides easy access to their scattering phase.

- We found that the semiclassical scattering phase reproduces, with the identification of charges of the scattering solitons as in equation (6.4), the complete bound state scattering matrix, incorporating both the fused BDS factor and the dressing phase. This corresponds to the first standpoint described in §4. Through the ’t Hooft coupling dependence of the fused BDS factor (which appears to be of a one-loop origin), this result provides an additional test of the integrability of the quantum world sheet theory. This is in the same spirit as the comparison of the one-loop world sheet corrections to the energies of extended semiclassical strings and finite size corrections from the gauge theory Bethe equations [35].

The equation (7.9) suggests that a possible approach to the calculation of the quantum corrections to the elementary magnon scattering matrix is to first find the quantum corrections to the large charge soliton-soliton scattering and discretize them. There are at least three seemingly different ways to approach such a calculation. On the one hand, one may use a moduli space approximation. To this end it is necessary to construct the sigma model scattering solution (perhaps along the lines of [15]), find the quantum corrections to the moduli space metric and then read off the corrections to the scattering amplitude. This approximation is clearly restricted to the small momentum limit. On the other hand, one may find the 1PI effective action of the world sheet sigma model and then simply search for solutions of its equations of motion which reduce in the classical limit to the initial scattering solutions and then use the WKB approximation. Alternatively, one may try to reformulate the world sheet sigma model such that the new fields correspond to solitons.¹⁵ Then, in this theory the quantum corrections we are interested in would be responsible for the double-poles in the bound state scattering matrix. It would be interesting to see if these approaches lead to consistent answers which are in agreement with those of [4].

- We have also found that the dressing phase by itself has a sigma model interpretation, the precise details of which depend on one’s choice of identification of soliton charges with CsG parameters. On the one hand, by making use of the ambiguity in the identification of the sigma model angular momenta J_2 and CsG parameters and

¹⁵In the case of the (real) sine-Gordon theory this reformulation is the Thirring model.

choosing a different identification for the two solitons, it is possible to adjust them such that the dressing phase (without the fused BDS contribution) is reproduced by the sigma model scattering. On the other hand, if we universally fix the identification of the sigma model angular momenta J_2 and CsG parameters as in [13], then the dressing phase is reproduced by the sigma model soliton-*antisoliton* scattering. These two observations are in fact related by the inability to differentiate between an isolated soliton and an isolated anti-soliton.

These observations, independently confirmed in the small momentum limit, follow the second standpoint described in §4 that, based on their 't Hooft coupling dependence, the fused BDS S-matrices should be considered of a quantum origin even though their J_2 dependence makes them scale semiclassically. It would be interesting to explicitly check whether quantum corrections to the classical soliton-antisoliton scattering indeed reproduce the contribution of the fused BDS phase. Since the soliton-soliton and soliton-antisoliton scattering matrices are related — in the complex sine-Gordon theory — by the relativistic crossing transformations, the result of such a comparison would give further information on the realization of crossing symmetry in the AdS/CFT correspondence.

Though it is included in our discussion, we have not separately analyzed the comparison of the scattering phase of a unit charge magnon off a fixed \mathcal{J}_2 bound state. The fact that one of the states has a semiclassical interpretation may make more tractable the calculation of quantum corrections to this scattering process. More generally, quantum corrections to world sheet solitons appear to be of importance for their relation to the gauge theory side of the AdS/CFT correspondence. It is possible that quantum corrections to the Hofman-Maldacena soliton will teach us about their scattering off “elementary” magnons.

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